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Warped Compactifications in M and F Theory

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Abstract

We study M and F theory compactifications on Calabi-Yau four-folds in the presence of non-trivial background flux. The geometry is warped and belongs to the class of p -brane metrics. We solve for the explicit warp factor in the orbifold limit of these compactifications, compare our results to some of the more familiar recently studied warped scenarios, and discuss the effects on the low-energy theory. As the warp factor is generated solely by backreaction, we may use topological arguments to determine the massless spectrum. We perform the computation for the case where the four-fold equals $K3 \times K3$.

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1 Introduction

Recently, there has been renewed interest in the idea that our world may be a domain wall in a higher dimensional spacetime [1]. This scenario, often known as the “Brane World” scenario [2, 3, 4, 5, 6, 7], is motivated by the existence of solitonic objects in string theory, such as D-branes, in which gauge and matter fields are localized on lower-dimensional subspaces. Indeed, if the Standard Model fields are localized on a brane, whereas gravity propagates in a higher dimensional spacetime, a number of phenomenological issues have to be revisited. A rather interesting 5-dimensional example was suggested in [8] where gravity seems to be localized on a 4-dimensional “brane”, even though the 5-th dimension has an infinite extent. A key ingredient in their setup is that the ambient spacetime is warped; the localization of gravity is due to an exponentially damped warp factor. Several attempts have been made to find domain wall solutions in 5D gauged supergravity that may perhaps localize gravity [9]. A warped metric can also be used to generate the hierarchy between the electroweak and the four-dimensional Planck scale [8]. The possible embedding of such a scenario in string theory was discussed in [10].

In fact, warped compactifications are a rather logical generalization of compactifications of the product type $\mathcal{M}_4 \times \mathcal{K}$, and were considered some time ago in the context of supergravity [11] and heterotic string theory [12, 13]. In perturbative string theory or supergravity there are no branes and in these scenarios there are therefore no gauge and matter fields which are a priori localized in some lower dimensional subspace. In strongly coupled heterotic string theory, on the other hand, there are two “end of the universe”- branes at the ends of the non-perturbative 11-th direction on which gauge and matter fields are localized [2, 6]. In general, the four-form field strength G is non-zero, and as a result, spacetime is warped. However, the long wavelength expansion in eleventh dimensional supergravity is valid provided that $R_{11} \leq R_{CY}/\kappa^{2/3}$ where κ is the eleventh dimensional gravitational coupling, and R_{CY} is the size of the Calabi-Yau on which the heterotic string theory is compactified. One cannot make the extra dimension transverse to the branes too large, or else the effective field theory breaks down.

F theory [14] provides another powerful tool in studying non-perturbative string vacua (see, *e.g.*, [15], for four-dimensional F theory compactifications). In compactifying M and F theory to lower dimensions, anomaly cancellation (cancellation of tadpoles) often requires one to introduce branes [18] (or alternatively, to turn on some background flux [16, 17]). All p -brane supergravity solutions are examples of warped metrics due to the backreaction of the branes. The recent developments give us motivation to better understand these warped compactifications in a variety of circumstances. And while the consistency requirements for M theory compactifications on Calabi-Yau four-folds (and their lifting to F theory) have been studied in some detail more recently [16, 17, 18, 19, 20], the form of the background geometry has not been determined explicitly in the presence of non-zero background flux (see, however [38]). A purpose of the present paper is to take some modest steps in this direction.

In the first part of this paper, we will determine the warp factor for some M/F theory compactifications with background flux and analyze its behavior. To solve the equation for the warp factor explicitly, we consider orbifold limits of Calabi-Yau spaces. In the F theory limit, the dilaton will also be taken to be constant over the base (the 7-brane charges are

locally cancelled). It is, however, likely that the generic features we find here will also hold for general Calabi-Yau manifolds. We find that the background flux contributes to the warp factor in the same manner and with the same scale as p -branes (both are α' suppressed as they are due to backreaction). The resulting metrics are thus generalizations of the p -brane ones with a contribution to the harmonic function from the background flux. In particular the warp factor always contains a constant part. This is to some extent expected as the contribution of the background flux to the energy momentum tensor scales inversely with the compactification volume. In the large volume limit, the contribution is negligible, and we should recover the regular p -brane metric. By way of comparison, the contribution of a cosmological constant (as in [8]) does not scale with the compactification volume. In the proposal of [10], however, one restricts attention to the near-horizon region where the constant part of the warp factor is unimportant and the effective geometry changes. The compactification data as well as the uncoupled gravitational interactions are encoded in a hypothetical Planck brane at the edge of the AdS throat.

In the second part of this paper we will address some aspects of the computation of (the closed string part of) the low-energy spectrum for compactifications with background flux. Generic string compactifications suffer from an abundance of massless fields. The presence of background flux, however, will lift a number of these [17]. *A priori*, the number of massless fields are determined by the moduli of the Calabi-Yau four-fold, subject to a set of topological consistency conditions on the non-zero background flux. On the other hand, a Kaluza-Klein reduction of an arbitrary warped metric generates extra mass terms due to the warp factors. Generically the massless spectrum is no longer determined purely from the dimensions of spaces of harmonic forms on the internal space. However, the powers of the warp factors that appear in this type of warped compactifications, which are determined by supersymmetry, balance each other out, so one may indeed use topological information to determine the massless spectrum. Moreover the non-trivial parts of the warp factor are generated solely by the gravitational backreaction, and in the weak coupling limit space-time reduces back to a product form. Using this information we recover explicitly the superpotential for the complex structure moduli conjectured in [20].

The organization of the paper is as follows. In Section 2, we review the results of M and F theory compactifications on Calabi-Yau 4-folds in the presence of background fluxes. In Section 3, we determine the warp factor for orbifold examples, in which the anomaly is cancelled by a combination of background flux (from the untwisted and twisted sectors) and $2/3$ -branes. In Section 4, we comment on the shape of the graviton wavefunction in this type of warped compactification. For completeness, we include in the appendix a parallel discussion on the graviton wavefunction for Heterotic-M theory on a Calabi-Yau 3-fold. In Section 5, we present a general discussion on determining the number of complex and Kahler moduli in vacua with background flux. For illustrative purpose, we consider in Section 6 compactifications of M theory with background flux which satisfy the field equations for an orbifold limit of $K3 \times K3$. The conditions on the background flux are greatly simplified if we choose it to be a product of $(1,1)$ forms of each $K3$. For this example we will determine the spectrum using the explicit topological data and we will discuss how to solve more general cases. We end with some comments in Section 7.

2 M and F theory Vacua with Background Fluxes

Warped compactifications of perturbative string theories, or rather supergravities, preserving minimal supersymmetry in four dimensions were considered in [12, 13]. The supersymmetry requirements are highly restrictive; it was found that for Type II theories there are no non-trivial solutions if the four non-compact dimensions are Minkowski [13]. For type I/heterotic theory there is a solution if the internal space has nonzero torsion: *i.e.* the vacuum expectation value of the three-form NS-NS field strength $\langle H^{NS} \rangle$ is non-zero. In this case the warp factor equals the dilaton, whose profile is determined by the vacuum expectation value $\langle H^{NS} \rangle$.

It has since been discovered that the M theory effective action has at the first subleading order in the derivative expansion a topological term [22]

$$- \int C \wedge X_8(R) , \quad (1)$$

where

$$X_8 = \frac{1}{8 \cdot 4!} \left(\text{tr} R^4 - \frac{1}{4} (\text{tr} R^2)^2 \right) . \quad (2)$$

The inclusion of such higher derivative terms in the low energy supergravities allows for new solutions preserving minimal supersymmetry, *i.e.* four supercharges, which are of the warped kind [16, 17]. (see also [18, 19, 20]). Let us give a brief review.

On compactifications down to three dimensions, the eight-form $X_8(R)$ can take on a background expectation value. This will act as a source term for the three-form field C . To maintain a solution to the field equations one needs to introduce M2-brane sources or non-trivial G -flux; G is the four-form which locally equals dC . Either the M2-branes or the non-trivial G -flux or both induce in turn a warping of the metric [16]

$$ds^2 = e^{-\phi(y)} \eta_{\mu\nu} dx^\mu dx^\nu + e^{\frac{1}{2}\phi(y)} g_{a\bar{b}} dy^a d\bar{y}^{\bar{b}} . \quad (3)$$

The requirement that minimal supersymmetry (four supercharges, $\mathcal{N} = 2$ in $d = 3$) is preserved determines the relative weights of the warp-factors, as is known from p -brane solutions [23]. In addition, supersymmetry demands that $g_{a\bar{b}}$ is the metric of a Calabi-Yau four-fold. The four-form G must furthermore obey

$$G_{abcd} = 0 = G_{abcd} , \quad g^{cd} G_{abcd} = 0 , \quad G_{\mu\nu\rho a} = \epsilon_{\mu\nu\rho} \partial_a e^{-\frac{3}{2}\phi} . \quad (4)$$

The first two conditions mean that G is a (2,2) form on CY_4 which is self-dual or, equivalently, primitive with respect to the Kahler form J of the CY_4 [17]

$$G = *G \quad \Leftrightarrow \quad J \wedge G = 0 . \quad (5)$$

Dirac quantization requires G to be an element of integer cohomology. The last equation of (4) says that the only nonvanishing part of the three-form C is

$$C_{\mu\nu\rho} = \epsilon_{\mu\nu\rho} e^{-\frac{3}{2}\phi} . \quad (6)$$

By definition the antisymmetric tensor is taken with respect to the unwarped metric.¹

The field equation for C determines the warp factor [16, 23]

$$\square_{CY_4}(e^{3\phi/2}) = *_{CY_4} \left(X_8 - \frac{1}{2} G \wedge G \right) - \sum_{j=1}^n \delta^8(y - y_j) . \quad (7)$$

The factor $G \wedge G$ has its origins in the well known $C \wedge G \wedge G$ supergravity interaction and we have introduced n M2-branes at arbitrary points. This number n is determined by the consistency requirement - the absence of tadpoles or anomalies - that the integral over the right hand side of (7) vanish,

$$\int_{CY_4} X_8 = \frac{\chi_{CY_4}}{24} = n + \frac{1}{2} \int_{CY_4} G \wedge G . \quad (8)$$

Lift to F theory

Such compactifications can be lifted to F theory if the CY_4 is elliptically fibered. Four-dimensional Lorentz-invariance requires that the four-form G has one leg on the toroidal fiber and three on the base \mathcal{B} [17]. We will limit our attention to the case where the G -flux is not localized in the fiber of the elliptic fibration, but is constant. In that case the M theory solution can straightforwardly be lifted to F theory with [17]

$$\begin{aligned} ds^2 &= e^{-\frac{3}{4}\phi} dx^2 + e^{\frac{3}{4}\phi} g_{a\bar{b}}^{\mathcal{B}} dy^a d\bar{y}^{\bar{b}} , \\ H^{NS} &= \omega - *_B \omega , \\ H^{RR} &= \omega \tau - *_B \omega \bar{\tau} , \\ D_{\lambda\mu\nu\rho}^+ &= \epsilon_{\lambda\mu\nu\rho} e^{-\frac{3}{2}\phi} , \end{aligned} \quad (9)$$

where $\omega \in H^{(1,2)}(\mathcal{B})$ with $(\tau - \bar{\tau}) \int_{\mathcal{B}} * \omega \wedge \omega = \int_{CY_4} G \wedge G \in \mathbb{Z}^+$. The metric and four-form potential are again that of the D3-brane solution. Formally the warp factor is still given by (7), but one has to be careful with the dependence on the internal directions. We will solve for the warp factor for a four-fold of the form $K3 \times T^4/\mathbb{Z}_2$ where the fiber belongs to T^4/\mathbb{Z}_2 . In this case, the warp factor can equivalently be determined from the IIB field equation for D^+ ,

$$d * dD^+ = \frac{1}{16} \sum_{i=1}^4 \text{tr}(R \wedge R) \delta^2(z^1 - z_i^1) - H^{NS} \wedge H^{RR} - *_B \sum_{j=1}^n \delta^2(z^1 - z_j^1) \delta^4(\omega - \omega_j) . \quad (10)$$

Here, i labels the fixed points of T^2/\mathbb{Z}_2 and j labels the n D3-branes. Tadpole cancellation requires that this number equals

$$\int_{\mathcal{B}} H^{NS} \wedge H^{RR} + n = \frac{\chi_{K3 \times K3}}{24} = 24 . \quad (11)$$

¹The three-form thus equals the induced volume on the three-dimensional space

$$C_{\mu\nu\rho} = \sqrt{\det |g_{MN} \partial_\mu z^M \partial_\nu z^N|} \epsilon_{\mu\nu\rho}^{Levi-Civita}$$

as is again familiar from p -branes. For the Dirac-delta function in curved space, we use the convention that $\int dx \sqrt{g(x)} \delta(x - x_i) = 1$.

Substituting the background expectation value for D^+ we find the F theory analogue of (7)

$$\square_{\mathcal{B}}(e^{3\phi/2}) = *_B \left(\frac{1}{16} \sum_{i=1}^4 \text{tr}(R \wedge R) \delta^2(z^1 - z_i^1) - H^{NS} \wedge H^{RR} \right) - \sum_{j=1}^n \delta^2(z^1 - z_j^1) \delta^4(\omega - \omega_j) , \quad (12)$$

which determines the shape of the warp factor. In the orientifold limit, the first term of the expression contains the contribution of the orientifold planes [26].

We will solve the warp-factor equations (7) and (12) for an orbifold limit of the Calabi-Yau. Specifically we will concentrate on the case $K3 \times K3 = T^4/\mathbb{Z}_2 \times T^4/\mathbb{Z}_2$. This allows us to take the F theory limit in the end, for which we will then attempt to compute the massless four-dimensional spectrum.

Let us note from the outset that the warp factor, as a solution to (7),(12) is determined only up to an integration constant whose value is fixed by the boundary conditions of the warp factor. If the fluxes are sufficiently localized, the metric (3) should be approximately flat away from such special points and the constant can be fixed to 1. We will discuss the effect of a constant flux in the next section.

The Dilaton

In the previously known warped compactifications of (heterotic) string theory, the warp factor turned out to be equivalent to the dilaton [12, 13]. Let us briefly recall why this is not the case for the Type IIB dilaton when we lift the above M theory vacua to F theory [17]. Weyl rescaling between the Einstein and the string frame can therefore alter the form of the warp factors if a non-constant dilaton profile is consistent with the field equations.

By compactifying M theory in the eleventh and ninth-direction to IIA on S_1 and T-dualizing on the latter one finds that the IIB dilaton is given by a ratio of the compactification radii [25]

$$\exp(\varphi_{IIB}) = \frac{R_{11}}{R_9} . \quad (13)$$

Since in the background solution for an elliptically fibered CY_4 ,

$$ds^s = e^{-\phi} dx^2 + e^{\phi/2} (g_{a\bar{b}} dy^a d\bar{y}^{\bar{b}} + R_9^2(y) dw_1^2 + R_{11}^2(y) dw_2^2) , \quad (14)$$

the warp factor is common to both R_{11} and R_9 , the IIB dilaton is independent hereof. In general, though, the dilaton will be a function of the internal dimensions y_a . In F theory, the 7-branes are sources of the complex field $\tau = a + ie^{-\varphi}$ where a is the RR axion. For example, near the location of a 7-brane at $z = z_i$ (z is the complex coordinate transverse to the 7-branes),

$$\tau \sim \frac{1}{2\pi i} \log(z - z_i) . \quad (15)$$

In the Einstein frame the metric is then warped to [28, 14]

$$g_{z\bar{z}} = \tau_2 \eta^2 \bar{\eta}^2 \prod_i (z - z_i)^{-1/12} \prod_i (\bar{z} - \bar{z}_i)^{-1/12} . \quad (16)$$

In F theory, the D7-branes are not mutually local, but in the orientifold limit, which we will consider, the charges of the D-branes are locally cancelled against the orientifold planes (which for non-zero g_s , are bound states of some (p, q) 7-branes). In that case the complex field τ becomes a constant, and the metric (16) reduces to a flat one (except for conical singularities at the locations of the orientifold planes). In the general case where the 7-brane charges are not locally cancelled, the calculation of the effective four dimensional Planck scale would involve an integral over a rather non-trivial function of the internal manifold (due to (15), (16)). Another complication is that the metric for the 3-7 brane system where the 7-brane charges are not locally cancelled has not yet been solved; for progress in this direction see [29].

3 The Warp Factor

M theory

The warp factor is to be determined from the equation of motion (7) for the gauge field $C_{\mu\nu\rho}$. To find its explicit form, we have to invert the Laplacian on the compact internal space. This can be done with the help of Green's functions. On a compact space, or equivalently when the Laplacian has non-trivial zero modes, the inversion is only defined on the subset of functions orthogonal to these zero modes, i.e. on those functions not belonging to the kernel of the Laplacian (see appendix for a brief review). For our purposes, it is relevant to know that on a compact space the scalar Green's function obeys,

$$\square G(x, x_i) = \delta^d(x - x_i) - \frac{1}{\text{Vol}} \quad (17)$$

where $\delta^d(x - x_i)$ is the d -dimensional Dirac delta-function, and "Vol" is the volume of the compact space.

On an orbifold, internal fluxes fall into two categories: those localized at the fixed points and those wrapped over periods of the underlying torus. The equation (7) determining the warp factor thus has two different kinds of contributions: the part of the flux term $G \wedge G$ which consists of a *constant* flux corresponding to "untwisted" cycles and that part which is built from "twisted" cycles, proportional to Dirac-delta-functions at the fixed points. The constant fluxes are naturally expressed as an integer divided by the volume (after the Hodge dual has been taken). In our case, the orbifold limit of $K3 \times K3$, we take the G-flux to be constant on one of the K3's, so that we may take the F theory limit later. Then the flux term on the r.h.s. of (7) can be written as

$$*(G \wedge G) = \frac{r_1 r_2}{V_1 V_2} + \frac{r_1}{V_1} \sum_{x_p} m_p \delta^4(x - x_p) , \quad (18)$$

where r_i , V_i is respectively the total untwisted flux and volume on each $K3$ and m_p is the total twisted flux located at each fixed point x_p of the second $K3$.

In the orbifold limit the curvature is also localized at the fixed points. On $K3 \times K3$ the total contribution to the warp factor from the curvature is

$$\int X_8 = \frac{\chi_{K3 \times K3}}{24} = 24 . \quad (19)$$

The $(T^4/\mathbb{Z}_2)^2$ limit has 16^2 fixed points. Each contributes of course equally to the curvature and the first term in (7) may thus be written as

$$*X_8 = \frac{24}{16^2} \sum_{x_p, z_p} \delta^4(x - x_p) \delta^4(z - z_p) , \quad (20)$$

where z_p are the fixed points of the first $K3$.

Combining this information the equation (7) determining the warp factor reduces to

$$\begin{aligned} \square(e^{3\phi/2}) &= \frac{3}{32} \sum_{x_p, z_p} \delta^4(x - x_p) \delta^4(z - z_p) - \frac{r_1 r_2}{2V_1 V_2} - \frac{r_1}{2V_1} \sum_{x_p} m_p \delta^4(x - x_p) \\ &\quad - \sum_{i=1}^n \delta^4(x - x_i) \delta^4(z - z_i) \\ &= \frac{3}{32} \sum_{x_p, z_p} \left(\delta^4(x - x_p) \delta^4(z - z_p) - \frac{1}{V_1 V_2} \right) - \sum_{i=1}^n \left(\delta^4(x - x_i) \delta^4(z - z_i) - \frac{1}{V_1 V_2} \right) \\ &\quad - \frac{r_1}{2V_1} \sum_{x_p} m_p \left(\delta^4(x - x_p) - \frac{1}{V_2} \right) . \end{aligned} \quad (21)$$

Here we have made use of the fact that tadpole cancellation implies that $48 = r_1 r_2 + r_1 \sum m_p + 2n$.

The warp factor is now easily expressed in terms of Green's functions

$$e^{3\phi/2} = c_0 + \frac{3}{32} \sum_{x_p, z_p} G^{(8)}(x, (x_p, z_p)) - \sum_{x_i, z_i} G^{(8)}(x, (x_i, z_i)) - \frac{r_1}{2V_1} \sum_{x_p} m_p G^{(4)}(x, x_p) . \quad (22)$$

Note that the completely constant flux term, proportional to r_2 , does not contribute to the warp factor in any obvious way. The Green's functions $G^{(8)}$ on $K3 \times K3$ and $G^{(4)}$ on a single $K3$ can be constructed in the orbifold limit by the method of images. For example, $G^{(4)}$ is given by

$$G^{(4)}(x, x_i) = - \sum_{\vec{p}} \frac{e^{i\vec{p}(\vec{x} - \vec{x}_i)} + e^{i\vec{p}(\vec{x} + \vec{x}_i)}}{\vec{p}^2} . \quad (23)$$

The momenta \vec{p} are quantized in units of the inverse radii of the T^4 . The second term is due to the \mathbb{Z}_2 image. In the above sum, the zero momentum mode ($p_1 = p_2 = p_3 = p_4 = 0$) is excluded.

The integration constant c_0 is determined by the boundary conditions. We already explained why, if all fluxes are localized ($r_2 = 0$ in our example above) this constant is fixed to unity. In that case far away from the points where flux (energy density) is localized, space-time should be approximately flat. In the case of constant G-flux over the whole internal manifold, we can determine c_0 from the curvature in the internal directions by looking at the Einstein equation,

$$R_{MN} = -\frac{1}{2} \left(G_M \cdot G_N - \frac{3}{D-2} g_{MN} G \cdot G \right) , \quad (24)$$

where

$$G \cdot G = \frac{G_{MNRS}G^{MNRS}}{4!} \quad ; \quad G_M \cdot G_N = \frac{G_{MABC}G_N^{ABC}}{3!} . \quad (25)$$

We have ignored contributions to the stress-tensor from the curvature and the M2-branes as they are localized. Substituting the background expectation values (and noting that G is self-dual in the internal dimensions)

$$C_{\mu\nu\rho} = \epsilon_{\mu\nu\rho} e^{-3\phi/2} \quad , \quad G_{\bar{a}\bar{b}\bar{c}\bar{d}} \equiv \tilde{G}_{\bar{a}\bar{b}\bar{c}\bar{d}} \neq 0 \quad \rightarrow \quad \tilde{G} \cdot \tilde{G} = r/V_{CY} , \quad (26)$$

one finds for the Ricci curvature in the internal directions²,

$$R_{mn} = \frac{1}{2} \left(\partial_m \ln e^{3\phi/2} \partial_n \ln e^{3\phi/2} - \frac{g_{mn}}{3} (\partial \ln e^{3\phi/2})^2 + g_{mn} \frac{e^{-3\phi/2}}{6} \tilde{G} \cdot \tilde{G} \right) , \quad (27)$$

with g_{mn} the metric on the CY_4 . In the decompactification limit the explicit G -flux term vanishes as it scales inversely with the volume. The warp factor does not depend explicitly on the constant flux and we should therefore recover the regular p -brane metric for which $c_0=1$. (Strictly speaking we have only shown that $\lim_{V_{CY} \rightarrow \infty} c_0(V) = 1$). The above argument is rather general, and is not restricted to orbifold cases where the curvatures are localized. This is again because $\int X_8$ and $\int G \wedge G$ are quantized (to a finite number), and so their contribution to the internal curvature is small when the size of the Calabi-Yau is large. In other words, both $\int X_8$ and $\int G \wedge G$, unlike the cosmological constant, are not extensive quantities. By way of comparison, a non-zero c_0 means that the decompactification limit of the present scenario has significantly different properties than those studied in [8]. For instance, one would not obtain a finite Planck scale in this limit.

An issue regarding the solution (22) is that due to the opposite sign of the Green's functions corresponding to the curvature induced charge, the metric has a naked singularity at the fixed points of the orbifold: the conical orbifold singularity. This is, however, an artifact of the long-range supergravity approximation. The solution may roughly only be trusted as long as the distance to any special point is larger than the Planck length or string scale. The naked singularity is expected to be cured by stringy effects. Examples of this have recently been discussed in [36, 37]. Since these effects will only modify the metric close to the singularity, they are unlikely to change the arguments above.

F theory

For F theory the solution is similar. We denote V_6 as the volume of the 6 compactified dimensions and V_2 the volume of the two dimensions transverse to the 7-branes. The twisted fluxes are proportional to $\delta^4(x - x_p)$ since we consider only fluxes that are not localized around a singular fiber of the elliptic fibration: constant on one of the K3's. The flux factor

²To find this answer one needs to use that the field equations require that

$$\tilde{G}_m \cdot \tilde{G}_n = \frac{g_{mn}}{2} \tilde{G} \cdot \tilde{G} .$$

This is the combined constraint of primitivity and $G \in H^{(2,2)}$ expressed in components.

can thus be written as

$$*(H^{NS} \wedge H^{RR}) = \frac{r}{V_6} + \frac{1}{V_2} \sum_{x_p} m_p \delta^4(x - x_p) . \quad (28)$$

The curvature contribution to the equation for the warp factor is due to the (unit charge) D3-branes or O3-planes (charge μ_O). In the orbifold limit the curvature is completely localized at the fixed points and thus gives rise to a six-dimensional delta-function. This corresponds to the fact that we are considering limit that the D7-brane charges are locally cancelled and we are left with only O3-plane charge at the fixed points. Explicitly

$$\square e^{\frac{3}{2}\phi} = \sum_{x_O} \mu_O \delta^6(x - x_O) - *(H^{NS} \wedge H^{RR}) - \sum_{i=1}^n \delta^6(x - x_i) . \quad (29)$$

Substituting (28) we can rewrite this as

$$\begin{aligned} \square e^{\frac{3}{2}\phi} &= \sum_{x_O} \mu_O \delta^4(x - x_O) \delta^2(z - z_O) - \frac{r}{V_6} - \frac{1}{V_2} \sum_{x_p} m_p \delta^4(x - x_p) - \sum_{i=1}^n \delta^6(x - x_i) \\ &= \sum_{x_O} \mu_O \left(\delta^4(x - x_O) \delta^2(z - z_O) - \frac{1}{V_6} \right) - \frac{1}{V_2} \sum_{x_p} m_p \left(\delta^4(x - x_p) - \frac{1}{V_4} \right) \\ &\quad - \sum_{i=1}^n \left(\delta^6(x - x_i) - \frac{1}{V_6} \right) , \end{aligned} \quad (30)$$

where we have made use of the fact that anomaly cancellation implies $r = n + \sum_{x_p} (m_p + \mu_O)$. In terms of Green's functions the warp factor is:

$$e^{\frac{3}{2}\phi} = c_0 - \frac{1}{V_2} \sum_{x_p} m_p G^4(x, x_p) + \sum_{x_O} \mu_O G^6(x, x_O) - \sum_{i=1}^n G^6(x, x_i) . \quad (31)$$

For the more general case where the 7-brane charges are not locally cancelled, equation (10) (which is written in the string frame), should receive contributions from O3- and O7-planes and D3- and D7-branes plus an appropriate contribution from the dilaton which is no longer constant. The net effect is that, in the Einstein frame, the warp factor is given by an equation of the form

$$\partial_i (\sqrt{g} g^{ij} \partial_j e^{3\phi/2}) = \sum_{x_O} \mu_O \delta^6(x - x_O) - H^{NS} \wedge H^{RR} - \sum_{i=1}^n \delta^6(x - x_i) , \quad (32)$$

where the metric g_{ij} takes into account the backreaction of the 7-branes according to (16). The exact solution to this equation is not known, although it can be solved approximately [29].

4 The Shape of Gravity

With the warped metric (9) in F theory, the tree-level four-dimensional Planck scale is given in terms of the 10-dimensional Planck scale by

$$M_4^2 = M_{10}^8 \int_B e^{-2\varphi} e^{3\phi/2} \sqrt{g^{\mathcal{B}}} , \quad (33)$$

whereas the tree-level gauge coupling on a three-brane at $y = y_i$ has no warp factor contribution and is given as usual by

$$\frac{1}{g_{YM}^2(y_i)} = e^{-2\varphi(y_i)} . \quad (34)$$

As the non-trivial terms in the warp factor are solely due to backreaction, the usual relation between the four-dimensional Newton constant and the three-brane gauge coupling still holds at leading order.

At subleading order, the warp factor might come into play. However, an explicit calculation shows that this is not the case. First, let us consider the case where the dilaton is constant. Note that the power of the warp factor that appears in (33) is exactly equal to the one in the field equation (12). Since the Green's function may be written as

$$G(y, y_i) = - \sum_{\lambda \neq 0} \frac{\bar{f}_\lambda(y) f_\lambda(y_i)}{\lambda^2} , \quad (35)$$

where f_λ are orthonormal eigenfunctions of the Laplacian with eigenvalues λ , the four-dimensional Planck mass can be expressed as

$$M_4^2 = M_{10}^8 e^{-2\varphi} \left(\int d^6 y \sqrt{g^{\mathcal{B}}} - \sum_{n \neq 0} \frac{1}{\lambda^2} \int d^6 y' \sqrt{g^{\mathcal{B}}} \rho(y') f_\lambda(y') \int d^6 y \sqrt{g^{\mathcal{B}}} \bar{f}_\lambda(y) \right) . \quad (36)$$

The quantity ρ equals the charge density on the r.h.s. of (12),

$$\rho(y') = *_B \left(\frac{1}{16} \sum_{i=1}^4 \text{tr}(R \wedge R) \delta^2(y' - y_i) - H^{NS} \wedge H^{RR} \right) - \sum_{j=1}^n \delta^6(y' - y_j) . \quad (37)$$

Because the zero mode is explicitly excluded in the Green's function, the second term in (36) vanishes. This bears out our expectations. The warp factor does not change the relation between the four and ten-dimensional Planck scale.³

In [8], an exponentially decaying wavefunction of the graviton was used to generate the hierarchy between the electroweak scale and the four-dimensional Planck scale. The exponential decay is due to the fact that the wavefunction of the massless graviton is dressed by some powers of the warp factor. By way of comparison, let us also deduce the shape of the graviton wavefunction in the present setup by linearizing fluctuations about the background

³ If the dilaton is not constant (taking into account the 7-branes), the net effect is the modification of the field equation to (32) where the metric is expressed in the Einstein frame. The above argument based on the properties of the Green's function should also hold in this case.

warped metric in Section 3. We will see that although the massless spectrum is indeed unaffected, the masses of the KK modes will get dressed by powers of the warp factor. A few comments are in order. If the compactification is of the order of string scale, we expect, *a priori*, that the wavefunctions of the KK modes are no longer given by linearizing Einstein gravity since stringy effects may become important. However, in compactifying Heterotic-M theory on a Calabi-Yau three-fold [6] (the G flux in this case is generically non-zero), there is a regime in which the theory is effectively five-dimensional, the wavefunctions of some of the lower-lying KK modes (in the fifth dimension) can still be obtained by linearizing Einstein gravity. The analysis for Heterotic-M theory on CY_3 is very similar, so for completeness, we have included a derivation of the KK spectrum in the appendix.

For a metric of the form

$$ds^2 = e^{2A(y)} \eta_{\mu\nu} dx^\mu dx^\nu + e^{2B(y)} \bar{g}_{ab} dy^a dy^b, \quad (38)$$

the gravitational fluctuations are given by $\eta_{\mu\nu} \rightarrow \eta_{\mu\nu} + h_{\mu\nu}$ where $h_{\mu\nu}$ is small compared with $\eta_{\mu\nu}$.

From Einstein's equations, if we choose the gauge $\partial^\mu h_{\mu\nu} = 0$, the linear fluctuations can be shown to satisfy the covariant wave equation (see the discussion in the next section or *e.g.* [30]):

$$\frac{1}{\sqrt{g}} \partial_M (\sqrt{g} g^{MN} \partial_N h_{\mu\nu}) = 0, \quad (39)$$

where the indices $M, N = 0, 1, \dots, 9$ are raised and lowered with the warped background metric (in Einstein frame). For the warped metric in M and F theory that we considered (*i.e.*, $A(y)$ is proportional to $B(y)$), this reduces to

$$(e^{-2A(y)} \square_{Mink} + \square_{\bar{g}}) h_{\mu\nu} = 0. \quad (40)$$

Expanding $h_{\mu\nu}(x, y) = \psi(y) \hat{h}_{\mu\nu}(x)$, with $\square_x \hat{h}_{\mu\nu}(x) = m^2 \hat{h}_{\mu\nu}(x)$, we have

$$\square_{\bar{g}} \psi(y) = -m^2 e^{-2A(y)} \psi(y). \quad (41)$$

Hence the masses of the KK modes depend on the warp factor, which in turn depends on the locations of the branes.

For the massless graviton (*i.e.*, $m^2 = 0$), (41) always admit the solution $\psi(y) = \text{constant}$. The wavefunction for the graviton, however, should be properly normalized:

$$S \sim \int d^{10}x \sqrt{g} \partial_\lambda h_{\mu\nu} \partial^\lambda h_{\mu\nu} + \dots = \int dy e^{-2A(y)} \sqrt{g(y)} \psi^2(y) \cdot \int d^4x \partial_{\hat{\lambda}} \hat{h}_{\mu\nu}(x) \partial^{\hat{\lambda}} \hat{h}_{\mu\nu}(x) + \dots \quad (42)$$

where hatted indices are with respect to the unwarped metric. Therefore, the properly normalized wavefunction is

$$\Psi(y) = \left[e^{-2A(y)} \sqrt{g(y)} \right]^{1/2} \psi(y) = e^{-\varphi} e^{3\phi/4} [\det g_{ab}^B]^{1/4}, \quad (43)$$

which is the square root of the integrand in (33), as expected. For the orbifold examples that we discuss in Section 3, the dilaton φ and $[\det g_{ab}^B]^{1/4}$ are constant, with $e^{3\phi/2}$ given in

terms of the Green's functions, hence

$$\Psi(y) \sim \left[c_0 - \frac{1}{V_2} \sum_{x_p} m_p G^4(x, x_p) + \sum_{x_O} \mu_O G^6(x, x_O) - \sum_{i=1}^n G^6(x, x_i) \right]^{1/2}. \quad (44)$$

An alternative view of these warped compactifications was suggested in [10]. In extending the AdS/CFT correspondence to the full string theory (*i.e.*, without taking the scaling limits as in [21]), the closed string degrees of freedom including gravity are no longer decoupled from the effective theory on the world-volume of the branes. To account for the closed string degrees of freedom, one introduces into the AdS supergravity a hypothetical Planck brane with dynamical degrees of freedom representing these closed string modes. This hypothetical Planck brane is placed at the edge of the AdS throat created by the branes and effectively cuts off the radial AdS coordinate. In the AdS/CFT correspondence, distances from the brane correspond to energy scales in the worldvolume theory. Therefore, the Planck brane serves as an UV cutoff and quantum gravity effects become important as we get closer to the Planck brane.

In the case that the spacetime transverse to the branes is compactified, the information about the compactification geometry would then be encoded in vacuum expectation values of the excitations of the Planck brane. In principle one could derive this set-up by integrating out coordinate shells of constant warp factor (momentum shells on the world volume) extending beyond the throat of the AdS near horizon region.

5 The Low Energy Spectrum

The low energy spectrum of warped compactifications is naively different from that of product space compactifications where one may use topological information of the internal manifold to determine the massless spectrum. For example the equation of motion of a scalar field,

$$\frac{1}{\sqrt{g}} \partial_M \sqrt{g} g^{MN} \partial_N \phi(z) = 0, \quad (45)$$

when reduced on a generic warped metric of the form

$$g_{MN}(z) = \begin{pmatrix} e^{2A(y)} \tilde{g}_{\mu\nu}(x) & 0 \\ 0 & \bar{g}_{ab}(y) \end{pmatrix} \quad (46)$$

yields

$$(e^{-2A} \square_{\tilde{g}} + \square_{\bar{g}} + d \bar{g}^{ab} \partial_a A \partial_b) \phi(z) = 0. \quad (47)$$

Here d denotes the dimension of the internal manifold; D is the dimension of the ambient space. Redefining $\phi(z) = e^{-dA/2} \tilde{\phi}(z)$ one finds that the low energy modes descending from this field are determined by

$$\left(e^{-2A} \square_{\tilde{g}} + e^{-\frac{dA}{2}} \left[\square_{\bar{g}} - \frac{d}{2} \square_{\bar{g}} A - \frac{d^2}{4} (\partial A)^2 \right] \right) \tilde{\phi}(z) = 0. \quad (48)$$

The last two terms act as additional mass terms for the dimensionally reduced field. The massless modes are those in which the eigenvalue of the internal Laplacian cancels the terms descending from the warp factor. Naively one thus loses the power of topological arguments to determine the massless spectrum.

The particular warped solutions known in string theory/supergravity, the p -brane metrics, belong to a special class, however. The internal manifold itself is also multiplied by a warp factor which precisely compensates for the warp factor of the external space. For example, in the scalar field above, the effect of an extra warp factor on the internal space,

$$g_{MN} = \begin{pmatrix} e^{2A(y)} \tilde{g}_{\mu\nu}(x) & 0 \\ 0 & e^{-2bA} \bar{g}_{ab}(y) \end{pmatrix}, \quad (49)$$

changes the field equation for the low-energy modes ϕ to

$$(e^{-2A} \square_{\tilde{g}} + \square_{\bar{g}} + (d - b(D - d - 2)) \bar{g}^{ab} \partial_a A \partial_b) \phi(z) = 0. \quad (50)$$

This internal warp factor can cancel the additional mass terms in (48) if b is chosen appropriately: $b = d/(D - d - 2)$. This is in fact exactly the combination one finds for p -brane metrics in supergravity. Here, the physical reason is that the warp factor is solely due to the backreaction of the vacuum configuration. Indeed one can explicitly see from the solution that the non-constant terms in the warp factor are suppressed by powers of the gravitational coupling constant. In the limit where this vanishes the warp factor is trivial and the space is of the product form $M_4 \times CY_4$. This is another argument why also in the case with background fluxes, the integration constant equals unity. In fact, the “balancing” of the warp factors is essentially the reason why when we first quantize open strings with Dirichlet-boundary conditions, the gravitational backreaction of the D-branes does not change the corresponding massless closed string spectrum.

The argument that in “balanced” warp metrics the warp factor is to a large extent inconsequential as regards to the massless part of the low-energy spectrum holds irrespective of whether G -flux is present or not. The introduction of the latter does complicate the determination of the massless spectrum as in the presence of non-trivial background expectation values of matter fields the mass matrix of low energy modes is generically off-diagonal in terms of the original fields. Fortunately the fact that the allowed background fluxes are subject to topological conditions - G should be primitive and a (2,2) form - in addition to the topological interpretation of the moduli of the Calabi-Yau four-fold ought to allow one to also use topological methods to determine the massless spectrum with G -flux [17]. In general these low-energy modes could be a linear combination of the original fields, but their number can be determined by looking at the topological constraints.

To be specific, those Kahler moduli which spoil the primitivity condition $J \wedge G = 0$ are lifted as well as those complex structure moduli which fail to keep G a (2,2) class. In addition those Wilson lines which are not orthogonal to G : $C \wedge G \neq 0$ are lifted as well [17]. In principle one has hereby computed the massless spectrum. In practice, the first and the last constraint are readily solved but the counting of those complex structure deformations which keep G a (2,2) class is more involved. Consider a deformation of the complex structure. This is given by a coordinate transformation which is not holomorphic

$$z^i \rightarrow y^i(z, \bar{z}). \quad (51)$$

Infinitesimally the mixed and the pure deformations of the metric under an arbitrary coordinate transformation are

$$\begin{aligned}\delta g_{a\bar{b}} &= g_{a\bar{c}}\bar{\partial}_{\bar{b}}\bar{y}^{\bar{c}} + g_{\bar{b}c}\partial_a y^c + y^c\partial_c g_{a\bar{b}} + \bar{y}^{\bar{c}}\bar{\partial}_{\bar{c}}g_{a\bar{b}} , \\ \delta g_{ab} &= g_{a\bar{c}}\partial_b\bar{y}^{\bar{c}} + g_{\bar{b}c}\partial_a\bar{y}^{\bar{c}} \equiv g_{\bar{c}(a}\bar{\chi}_{b)}^{\bar{c}} .\end{aligned}\tag{52}$$

This is just the well known fact that non-holomorphic coordinate changes correspond to pure type metric deformations. By contracting $g^{\bar{c}a}\delta g_{ab} \equiv h_b^{\bar{c}}$ with the constant anti-holomorphic $(0, n)$ form one recovers the $(1, n-1)$ forms that are in one-to-one correspondence with complex structure deformations. Under such a transformation a $(2,2)$ form transforms infinitesimally as

$$G \equiv G_{a\bar{b}c\bar{d}}dz^a d\bar{z}^{\bar{b}} dz^c d\bar{z}^{\bar{d}} \rightarrow G + K^{(3,1)} + K^{(1,3)}\tag{53}$$

with

$$K^{(1,3)} = G_{a\bar{b}c\bar{d}}(\chi_e^a\delta_e^{\bar{c}} - \chi_e^{\bar{c}}\delta_e^a)dz^e d\bar{z}^{\bar{b}} d\bar{z}^{\bar{d}} d\bar{z}^{\bar{e}} .\tag{54}$$

For those complex structure deformations which keep G a $(2,2)$ class $K^{(1,3)}$ must vanish. This constraint can be expressed as follows. The complex structure deformations χ_b^a are representatives of the cohomology $H^{(0,1)}(T)$. Define the form $\tilde{G} \in H^{(0,2)}(\wedge^2 T)$

$$\tilde{G}_{\bar{c}\bar{d}}^{ab} = G_{\bar{c}\bar{d}ef}\Omega^{efab} .\tag{55}$$

The constraint that $K^{(3,1)}$ must vanish can then be expressed by requiring that the triple intersection numbers

$$\int \Omega_{abcd} \left(\chi_{(0,1)}^a \wedge \tilde{G}_{(0,2)}^{bc} \wedge \tilde{\alpha}_{I(0,1)}^d \right) \wedge \Omega_{(4,0)} = 0\tag{56}$$

vanish for all basis elements $\tilde{\alpha}_I$ of $H^{(0,1)}(T)$. The latter are constructed from basis elements α_I of $H^{(3,1)}(CY_4)$ as in (55). Equation (56) thus represents that the natural inner product of $K^{(1,3)}$ with an arbitrary element α_I vanish. There is however no general analysis of this condition if the $(2,2)$ form G is also required to be integral. One therefore has to solve (56) on a case by case basis. Below we will do so for the simple example where the $CY_4 = K3 \times K3$.

The deformations that do not respect the constraints should give rise to mass terms when we perform a KK reduction. By our previous reasoning we can to first order ignore the effects of the warp factor. Then, we are on a product space and the KK reduction is straightforward, with the caveat that the field equations are now not satisfied. One finds that the kinetic term for C is responsible for mass terms for the pure and mixed type metric fluctuations (numerical coefficients are suppressed)

$$m^2 h^2 \sim \int h^{ab} G_{a\bar{b}c\bar{d}} G_{b\bar{c}}^{\bar{c}\bar{d}} h^{\bar{b}\bar{c}} + h^{a\bar{b}} G_{a\bar{b}e\bar{d}} G_{\bar{b}\bar{c}}^{\bar{e}\bar{d}} h^{\bar{b}\bar{c}} + G_{a\bar{b}c\bar{d}} h^{\bar{d}\bar{k}} h_{\bar{k}}^e G_e^{\bar{a}b\bar{c}} + G_{a\bar{b}c\bar{d}} h^{\bar{d}\bar{k}} h_{\bar{k}}^e G_e^{\bar{a}b\bar{c}} .\tag{57}$$

In the first and the third term one recognizes the “square” of $K^{(3,1)}$ in (54). For massless deformations it must vanish.

One may compare this with the conjectures made in [20]. There it was argued that the complex structure deformations have a superpotential

$$W_C = \int \Omega \wedge G ,\tag{58}$$

where Ω is the holomorphic (4,0) form. For the calculation of the mass terms involving the complex structure deformations, the relevant parts of the first and second deformation of the four-form Ω are

$$\begin{aligned} (\delta\Omega)_{abc\bar{n}} &\sim \Omega_{abcd}g^{d\bar{m}}h_{\bar{m}\bar{n}} + \dots, \\ (\delta^2\Omega)_{ab\bar{n}\bar{f}} &\sim \Omega_{abcd}g^{c\bar{m}}h_{\bar{m}\bar{n}}g^{d\bar{e}}h_{\bar{e}\bar{f}} + \dots \end{aligned} \quad (59)$$

The non-trivial quadratic fluctuations in the superpotential are thus given by

$$\delta^2 W = \int \delta^2 \Omega \wedge G \sim \int \sqrt{g} h_{\bar{m}}^n h_{\bar{c}}^d \Omega_{abdn} G^{ab\bar{c}\bar{m}}, \quad (60)$$

where we have used the fact that the background value of $G^{(2,2)}$ is selfdual. Reducing to $N = 0$ components and noting that the $W^2|_{\theta=0}$ term does not contribute, we find the mass term for the pure deformations

$$m^2 h_{zz}^2 \sim \int h_{\bar{c}}^d \Omega_{abdn} G^{ab\bar{c}\bar{m}} G^{np\bar{q}\bar{r}} \bar{\Omega}_{\bar{r}\bar{q}\bar{s}\bar{m}} h_p^{\bar{s}}. \quad (61)$$

Using the relation

$$\Omega_{abcd} \bar{\Omega}_{\bar{a}\bar{b}\bar{c}\bar{d}} \sim \epsilon_{abcd\bar{a}\bar{b}\bar{c}\bar{d}}, \quad (62)$$

the mass term reduces to

$$m^2 h_{zz}^2 \sim \int h_{\bar{s}}^d G_{\bar{q}\bar{r}nd} G^{np\bar{q}\bar{r}} h_p^{\bar{s}} + h_{\bar{q}}^d G_{\bar{s}\bar{r}nd} G^{np\bar{q}\bar{r}} h_p^{\bar{s}}, \quad (63)$$

which is of the same form as the first and third mass terms from the Lagrangian in (57).

For the Kahler deformations the authors of [20] conjectured the superpotential

$$W_K = \int \mathcal{K} \wedge \mathcal{K} \wedge G, \quad (64)$$

where K is the complexified Kahler form $\mathcal{K} = J + iB$. The reduction to $N = 0$ components in this case is more involved. This will be reported elsewhere.

6 An Example: $K3 \times K3$

In this section, we will compute the spectrum for the simple example where the CY_4 equals $K3 \times K3$. We will first determine the number of complex and Kahler moduli by explicitly constructing the complete parameter space of solutions to the G -flux constraints on the orbifold limit of $K3 \times K3$. This allows us to simply count the number of moduli by hand. At the end we will compare it to the topological determination of allowed deformations outlined above.

Vanishing Fluxes

Before proceeding let us briefly recall for comparison the spectrum of M theory on a $K3 \times K3$ without G-flux. In this case there are $H^{(3,1)} + H^{(2,1)}$ $N = 1, d = 4$ chiral multiplets, corresponding to deformations of the complex structure and Wilson lines of the three-form respectively, and $H^{(1,1)}$ $N = 1, d = 4$ vector fields, corresponding to Kahler deformations. For $CY_4 = K3 \times K3$ the Hodge diamond equals

$$\begin{array}{cccccccccccc}
& & & & h^{0,0} & & & & & & 1 & & & \\
& & & & h^{1,0} & h^{0,1} & & & & & 0 & 0 & & \\
& & h^{2,0} & & h^{1,1} & h^{0,2} & & & & 2 & 40 & 2 & & \\
h^{3,0} & & h^{2,1} & h^{1,2} & h^{0,3} & & & 0 & 0 & 0 & 0 & & \\
h^{4,0} & h^{3,1} & h^{2,2} & h^{1,3} & h^{0,4} & 1 & 40 & 404 & 40 & 1 & & &
\end{array} = \quad (65)$$

The $K3 \times K3$ compactification is non-minimal in that it preserves four more supercharges than the minimal four required by $N = 1, d = 4$ supersymmetry. Indeed we see that the spectrum is given by 40 $N = 2, d = 4$ vector multiplets.

Taking one of $K3$'s to be elliptically fibered we can lift this M theory compactification to F theory on $K3 \times K3$, which is related to a number of other compactifications by a chain of dualities [18, 31, 32]. Since F theory on $K3$ is heterotic on T^2 , compactifying both sides on another $K3$ gives heterotic on $K3 \times T^2$. On the other hand, heterotic on $K3$ is F theory on a Calabi-Yau threefold CY_3 . As a result, F theory on $K3 \times K3$ is dual to F theory on $CY_3 \times T^2$.

The orientifold dual of the above F theory compactification can be found via Sen's map [33]. We take the orbifold limit $K3 = T^4/\mathbb{Z}_2$ for each $K3$. Let (z_1, z_2, z_3, z_4) be the complex coordinates of T^8 , with z_4 being the fiber coordinate. The orientifold dual is given by Type IIB on $T^6/\{R_1 \times (\Omega(-1)^{F_L} R_2)\}$ where Ω is the worldsheet parity operation, R_1 and R_2 act as follows:

$$R_1 z_{1,2} = -z_{1,2} \ , \quad R_1 z_3 = z_3 \ , \quad (66)$$

$$R_2 z_{1,2} = z_{1,2} \ , \quad R_2 z_3 = -z_3 \ . \quad (67)$$

The resulting orientifold dual is therefore the Gimon-Polchinski model [34, 35] further compactified on a T^2 and then T-dualized in the T^2 directions, so that instead of 5-branes and 9-branes, we have 3-branes and 7-branes. At a special point of the moduli space where τ is constant, *i.e.*, four $D7$ -branes are placed on top on each $O7$ -plane, the gauge group from the 7-branes is $U(4)^4$. In addition, there are two $\mathcal{N} = 2$ hypermultiplets in the **6** representation of each $U(4)$ from the 77 open strings. The 3-branes give rise to additional gauge symmetries. In the case of maximal gauge symmetries, *i.e.* the 3-branes are sitting on top of each other, the gauge group from the 3-branes is $U(16)$. The 33 open strings also give rise to two $\mathcal{N} = 2$ hypermultiplets in the **120** representations of $U(16)$. Finally, there is a hypermultiplet in the bi-fundamental representation (**16**, **4**) of $U(16) \times U(4)$ if the 3-branes sit on one of the four groups of 7-branes. The orientifold model at a generic point of the moduli space can be obtained from the above by Higgsing.

6.1 G-flux conditions

Now we turn on background fluxes. The field-equations demand that we seek an integer (2,2) form on $K3 \times K3 = T^4/\mathbb{Z}_2 \times T^4/\mathbb{Z}_2$ which is primitive, i.e.

$$J \wedge G = 0 . \quad (68)$$

We will choose G of the form

$$G = \omega_1 \wedge \omega_2 , \quad (69)$$

where $\omega_i \in H^{(1,1)}(T^4/\mathbb{Z}_2) \cap H^{(2)}(T^4/\mathbb{Z}_2, \mathbb{Z})$. This guarantees that $G \in H^{(2,2)}(T^4/\mathbb{Z}_2, \mathbb{Z})$ though it is not necessary. This simplifies the solutions to the quantization conditions (as compared to the general case [17]). We could have chosen $\omega_i \in H^{(2,0)}(T^4/\mathbb{Z}_2, \mathbb{Z})$ or $H^{(0,2)}(T^4/\mathbb{Z}_2, \mathbb{Z})$ but these forms correspond to the class of the fiber and its Hodge dual and are no longer normalizable when we shrink the volume of the fiber in the F theory limit [17].

The complex structure is inherited from the tori and the condition (68) requires that each ω_i is primitive with respect to J_i [17]. Thus our task is reduced to finding primitive (1,1) forms on T^4/\mathbb{Z}_2 . The cohomology $H^{(1,1)}(K3 = T^4/\mathbb{Z}_2)$ has dimension 20, but these can be subdivided in four “untwisted” (1,1) forms inherited from the T^4 and sixteen “twisted” ones. As any integer form is an integer combination of the basis forms we can consider the two situations separately.

Constant fluxes

Consider untwisted forms first. These are inherited from the T^4 and are constant on the orbifold. The intersection matrix of two-forms on $K3 = T^4/\mathbb{Z}_2$ is block-diagonal with in the upper left-hand corner minus the Cartan-matrix of $E_8 \times E_8$ and in the lower right-hand corner three times the Pauli matrix σ_1 . The latter blocks equal the intersection-matrix of two-forms on T^4 and thus correspond to the untwisted forms. Hence we only have to check that the periods of the untwisted forms are integer over the untwisted cycles. This means that we have reduced our problem to finding the set of integer (1,1) forms D on T^4 .

For each T^2 (with volume normalized to 1), we make the standard choice for the periods

$$\int_{\gamma_x^j} dz^i = \delta_j^i \quad \int_{\gamma_y^j} dz^i = \tau_i \delta_j^i \quad (\text{no sum on } i) , \quad (70)$$

where γ_x^i and γ_y^i are the x and y cycles; $i, j = 1, 2$. The (1,1) forms $\tilde{\gamma}_x^i$ and $\tilde{\gamma}_y^i$ dual to these cycles are

$$\tilde{\gamma}_x^i = dy^i \quad \tilde{\gamma}_y^i = -dx^i , \quad (71)$$

with the volume element

$$\int_{T_i} dx^i \wedge dy^i = 1 \quad (\text{no sum on } i) . \quad (72)$$

The Kahler form is given by

$$J_i = dz^i \wedge d\bar{z}^i , \quad (73)$$

and equals $(\bar{\tau}_i - \tau_i)$ times the volume form $dx^i \wedge dy^i$.

Take the Kahler form J for the T^4 to be the sum of J_1 and J_2 . The general form of (1,1) forms obeying the primitivity condition is then

$$J \wedge D = 0 \rightarrow D = Adz_1d\bar{z}_2 + \bar{A}d\bar{z}_1dz_2 + iB(dz_1d\bar{z}_1 - dz_2d\bar{z}_2) . \quad (74)$$

where A is complex and B is real. The requirement that $D \in H^{(1,1)}(T^4, \mathbb{Z}) \subset H^{(1,1)}(T^4/\mathbb{Z}_2, \mathbb{Z})$ demands that

$$\begin{aligned} A + \bar{A} &= n , \\ \bar{\tau}_2 A + \tau_2 \bar{A} &= m , \\ \tau_1 A + \bar{\tau}_1 \bar{A} &= p , \\ \tau_1 \bar{\tau}_2 A + \bar{\tau}_1 \tau_2 \bar{A} &= q , \end{aligned} \quad (75)$$

where $n, m, p, q \in \mathbb{Z}$. In addition

$$\begin{aligned} iB(\bar{\tau}_1 - \tau_1) &= v , \\ iB(\bar{\tau}_2 - \tau_2) &= w , \end{aligned} \quad (76)$$

where $v, w \in \mathbb{Z}$. These are six equations with in principle three unknowns: $\text{Re}(A)$, $\text{Im}(A)$, B ; the system is overconstrained and has no solutions.

However, if one relaxes the requirement that all the τ_i are free parameters, namely one allows the complex structure τ_2 of one torus to be determined in terms of the other, and in addition requires that the ratio of the imaginary parts of τ_1 and τ_2 be a rational number, then there is a solution. This poses three real constraints on the moduli of the T^4 . The first constraint can be seen by rewriting the integrality conditions as

$$\begin{aligned} \bar{A} &= n - A , \\ A &= \frac{p - n\bar{\tau}_1}{(\tau_1 - \bar{\tau}_1)} = \frac{m - n\tau_2}{(\bar{\tau}_2 - \tau_2)} , \\ \frac{\tau_1 - \bar{\tau}_1}{\tau_2 - \bar{\tau}_2} &= \frac{n\bar{\tau}_1 - p}{m - n\tau_2} , \\ \tau_1 \bar{\tau}_2 A + \bar{\tau}_1 \tau_2 \bar{A} &= q . \end{aligned} \quad (77)$$

Note, however, that we have imposed both the primitivity and integrality conditions in deriving these constraints. Moreover, in (74) we are only considering (1,1) classes. Hence, the three constraints determine the loci in the combined space of complex and Kahler structures on K3 where one may find primitive integral forms which are purely (1,1). The number of constraints therefore correspond to the total number of moduli, complex structure plus Kahler, which are lifted. Complex structure deformations must form complex pairs. The total number of moduli which become massive is therefore 1 (complex valued) complex structure deformation and 1 real Kahler deformation. For instance, one can see this explicitly by generalizing the choice of the Kahler class to $J_1 + aJ_2$ and then noting that the analogue of eq.(76) fixes the value of a in terms of the complex structures τ_1 and τ_2 .

In the end we are interested in the $H^{(2,2)}(T^4, \mathbb{Z})$ flux

$$\int_{T^4} D \wedge D = 2(|A|^2 - B^2)(\tau_1 - \bar{\tau}_1)(\tau_2 - \bar{\tau}_2) . \quad (78)$$

which should be an integer. This is proportional to the number of M2/D3-branes we will have to introduce. Substituting the second equation of (77) in (78) we find that

$$\int_{T^4} D \wedge D = -2vw - 2(p - n\bar{\tau}_1)(m - n\bar{\tau}_2) . \quad (79)$$

As the last factor should be a negative semi-definite integer ($\tau_i \in H^+$ in (78)) let us make the Ansatz that

$$m - n\bar{\tau}_2 = \frac{r}{(p - n\bar{\tau}_1)} \rightarrow \bar{\tau}_2 = \left(\frac{m}{n} - \frac{r}{n(p - n\bar{\tau}_1)} \right) \quad (80)$$

is indeed a solution to (77). Here $r \in \mathbb{Z}^+/2$. Substituting this relation into the four equations (77) we find that our Ansatz is a solution with $r = -qn + mp$. Note that $mp \geq qn$. Hence the flux equals

$$\int_{T^4} D \wedge D = -2vw - 2(p - n\bar{\tau}_1)(m - n\bar{\tau}_2) = 2(qn - mp - vw) . \quad (81)$$

Note that if D is also an element of $H^{(1,0)}(T^2, \mathbb{Z}) \times H^{(0,1)}(T^2, \mathbb{Z}) \subset H^{(1,1)}(T^4, \mathbb{Z})$, i.e. if $n = n_1 n_2$; $m = n_1 m_2$; $p = m_1 n_2$ and $q = m_1 m_2$ for integer n_1, n_2, m_1, m_2 then the flux $\int_{T^4} D \wedge D = 2(m_1 m_2 n_1 n_2 - n_1 m_2 m_1 n_2)$ vanishes.

Twisted fluxes

We will restrict ourselves to twisted forms whose dual cycles are localized at the fixed points. These are two-spheres $S^2 \simeq CP_1$ shrunk to a point. Denoting them as B^i with $i = 1, \dots, 16$ running over the fixed points, the general form of a twisted flux D is

$$D = c_i B^i . \quad (82)$$

The B^i are a linear combination of the generating forms V^n of $H^{(1,1)}(T^4/\mathbb{Z}_2, \mathbb{Z})$: $B^i = a_n^i V^n$ with a_n^i integer. As we are limiting our attention to twisted fluxes we may restrict the V^n to those generating minus the $E_8 \times E_8$ Cartan matrix I^{nm} . This also means that D is automatically primitive as the Kahler form J consists purely of untwisted forms.

The condition that also $D \in H^{(1,1)}(T^4/\mathbb{Z}_2, \mathbb{Z})$ means that

$$\int_{V^n} D = -c_i a_m^i I^{mn} = p^n , \quad (83)$$

with $p^n \in \mathbb{Z}$ for all n . The cycles B^i have an intersection matrix [12]

$$\int B^i \wedge B^j = -2\delta^{ij} = -a_n^i I^{nm} a_m^j . \quad (84)$$

This means that the coefficients c_i are all multiples of $1/2$,

$$c^i = -\frac{p^n a_n^i}{2} . \quad (85)$$

The four-form flux of interest to us $\int_{T^4/\mathbb{Z}_2} D \wedge D$ equals

$$\int_{T^4/\mathbb{Z}_2} D \wedge D = -2 \sum_i c^i c^i = -p^n I_{nm}^{-1} p^m, \quad (86)$$

as one can show that

$$\sum_i a_n^i a_m^i = 2 I_{nm}^{-1}. \quad (87)$$

The inverse of the Cartan matrix has integer entries because the E_8 lattice is even and self-dual, and the diagonal entries are even. The r.h.s. of (86) is therefore an integer as expected. Note again that $\int D \wedge D$ is negative semidefinite just as in the case of constant fluxes.

Note that the B_i or more precisely arbitrary half-integer combinations thereof do not generate $H^{(1,1)}(T^4/\mathbb{Z}_2, \mathbb{Z})$ [12], but a larger group. For integer cohomology the c^i are subject to the additional constraint

$$\sum_i a_m^i c^i = p^n I_{nm}^{-1}. \quad (88)$$

We are therefore not missing any integer forms by limiting our attention to fluxes of the form (82).

In this case those Kahler deformations $\delta J = \beta_n V^n$ where $V^n \in H^{(1,1)}(T^4/\mathbb{Z}_2, \mathbb{Z})$ are twisted fluxes that do not preserve the primitivity condition are frozen. Those deformations such that

$$\int \delta J \wedge D = -\beta_n I^{nm} c_i a_m^i = 0 \quad \forall \beta_n \quad (89)$$

do survive. The integrated primitivity condition is sufficient to guarantee the local one [12].

In fact we can determine the contribution to $\int D \wedge D$ of each fixed point separately. As the intersection matrix of the cycles B^i is proportional to the unit matrix, the contribution of the ' i 'th cycle is just given by the ' i 'th term in equation (86) [12]. Since the orientifold planes are also localized at the fixed points twisted fluxes change the effective O-plane charge.

6.2 The Spectrum

In the previous section, we explicitly constructed the parameter space of solutions for pure $(1,1)$ forms on $K3$ that are primitive and integral. For constant flux this showed that the dimension of the space of complex structures and the dimension of the space of Kahler forms are each reduced by one. In this section we will compare these results with a topological deformation analysis.

Suppose one is given a complex structure and Kahler form on $K3$ for which it is possible to find a primitive integral $(1,1)$ flux ω_i . The $\omega_i \in H^{(1,1)}(K3) \cap H^2(K3, \mathbb{Z})$ define a single direction in $H^{(1,1)}(K3) \cap H^2(K3, \mathbb{Z})$. The Kahler deformations which are preserved are those which are orthogonal to ω_i in the sense that

$$\int \delta J \wedge \omega_i = 0. \quad (90)$$

Thus there is always exactly one Kahler deformation in each $K3$ that is lifted.

As for the complex structure deformations, on a $K3$ they will cause a $(1,1)$ form to become a mixture of a $(1,1)$ and a $(0,2)+(2,0)$ form; the latter being complex conjugates. Since $H^{(0,2)}$ is 1 dimensional, a 19 dimensional subspace of $(1,1)$ forms is preserved. These allowed deformations will not spoil integrality, but could spoil primitivity. One can, however, always find a compensating Kahler deformation which restores primitivity. This is illustrated in the constant flux example by our earlier explicit calculation.

Since on $K3 \times K3$, the $(2,2)$ classes come from $(1,1) \times (1,1)$ and $(2,0) \times (0,2)$, we find that there is a $38 = 19 + 19$ dimensional subspace of preserved complex structure deformations on $K3 \times K3$.

An alternative way to see this embodies the symmetry between complex structure deformations and Kahler deformations. Consider in analogy with the superpotential (58) of [20] the intersection

$$\int \Omega_i \wedge \omega_i \quad (91)$$

for each $K3$. Ω_i is now the holomorphic $(2,0)$ form. Under a complex structure deformation this potential changes to

$$\int \delta\Omega_i \wedge \omega_i . \quad (92)$$

We see that that deformation which is parallel to the ω_i flux is lifted.

These constraints thus change the number of chiral and vector multiplets by one in each $K3$. As a result, a total of two $N = 2$, $d = 4$ vector multiplets are lifted due to the G-flux. The spectrum (in addition to that from the branes) thus consists of 38 $N = 2$, $d = 4$ $U(1)$ vector multiplets, coupled to gravity.

7 Discussion

In this paper, we have taken some modest steps towards understanding warped compactifications in M and F theory on Calabi-Yau four-folds in the presence of non-trivial background flux. The introduction of background flux can have interesting consequences for low energy physics and this subject is certainly worthy of further study. Detailed investigations require the explicit form of the warp factor, and to facilitate its determination, we considered orbifold limits of these compactifications. The contribution of the background flux is a backreaction effect similar to explicit p -brane sources and because the energy density associated with the G-flux is inversely proportional to the volume, the leading term in the warp factor is always a constant. This contrasts sharply to what has been found in a number of other warped scenarios. In the orbifold limit the background fluxes are either constant or localized. The constant fluxes allow the warp factor to be consistently solved in terms of Green's functions on the internal space. The twisted fluxes act as sources for the Green's functions.

The extensive nature of the background flux suggests that its introduction will not have drastic consequences; indeed we find that the usual relation between the four- and ten-dimensional Planck scales is recovered. This, again contrasts with the scenarios involving a cosmological constant [8], in which gravity is localized. The shape of the graviton wavefunction is sharply peaked at the location of the branes but this is just the source divergence of a Green's function. In a similar analysis for Heterotic-M theory on a Calabi-Yau three-fold,

discussed in an appendix, we also derived the wavefunctions of the KK modes. In our M/F theory setup, the higher KK modes can also be obtained by linearizing gravity if the compactification size is large. Interestingly, aside from the compactification size, at subleading order the masses of the KK modes appear to depend on the location of the branes.

In the view of [10], in which the renormalization flow of the effective world volume theory is correlated with a translation in one of the real internal directions, the compactification geometry is encoded by a hypothetical Planck brane at the throat of the near-horizon AdS created by the branes. The shape of the warp factor should determine the characteristics of this Planck brane.

Finally we discussed the low energy spectrum of these warped compactifications. Supersymmetry or the argument that the warp factor is solely due to backreaction guarantees that topological arguments may still be used to determine the massless spectrum. The presence of background fluxes can lift a number of moduli, but the rules to determine them are of (pseudo) topological nature as well. For $K3 \times K3$ we found that two $N = 2$ vector multiplets were lifted. In addition to the complex and Kahler moduli that arise in these compactifications, the low energy degrees of freedom include gauge and matter fields on the branes. For example, in the $K3 \times K3$ case, there are generically 3 and 7 branes on which gauge and matter fields are localized. We recalled in Section 6 the spectrum from the branes when the background flux is zero. The determination of the spectrum from the brane sector of F theory compactifications in the presence of background three-form NS-NS and R-R fluxes is an interesting direction to pursue in the future.

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Appendix

A. Green's Functions on a compact space

On a compact space, or equivalently when the Laplacian has non-trivial zero modes, the inversion of the Laplacian is only defined on the subset of functions orthogonal to these zero-modes. One can see this explicitly by constructing the Green's function in terms of the complete set of eigenfunctions of the Laplacian. These eigenfunctions must obey the same boundary conditions as the function on which the Laplacian acts. Recalling that the Laplacian has non-positive definite eigenvalues we can denote an orthonormal set by $\phi_m(x)$ with

$$\square \phi_m(x) = -m^2 \phi_m(x) . \quad (93)$$

The Green's function is then

$$G(x, x') = - \sum_{m \neq 0} \frac{\bar{\phi}_m(x) \phi_m(x')}{m^2} . \quad (94)$$

All the zero modes of the Laplacian must be omitted in the sum on the r.h.s. This is a generic feature of all Green's functions. When one writes

$$\square G(x, x') = \mathbb{1} = \text{“}\delta^D(x - x')\text{”} , \quad (95)$$

it is implicitly understood that the δ -function or identity-operator is only defined in the space of functions orthogonal to the zero-modes of the Laplacian. This is not necessarily the same as the Dirac-delta function. On a circle of radius R for instance, there is exactly one non-trivial zero-mode, the constant function. The Green's function equals

$$G(x, x') = - \frac{1}{2\pi R} \sum_{n \neq 0} \frac{e^{\frac{in(x-x')}{R}}}{n^2/R^2} \quad (96)$$

and therefore obeys⁴

$$\square G(x, x') = \partial_x^2 G(x, x') = \frac{1}{2\pi R} \sum_{n \neq 0} e^{\frac{in(x-x')}{R}} = \delta(x - x') - \frac{1}{2\pi R} , \quad (97)$$

rather than

$$\square G(x, x') = \delta(x - x') . \quad (98)$$

The former combination $\delta(x - x') - \frac{1}{2\pi R}$ is obviously the delta-function in the space of functions orthogonal to the zero-mode. In the infinite volume limit we are left with the conventional Dirac-delta function.

The above is a reflection of the fact that for differential forms, the Green's function can be used as a projector onto the space of non-harmonic forms, see e.g. [27]. An arbitrary p -form ω can be decomposed into an harmonic p -form γ and a non-harmonic part as

$$\omega = \gamma + \square G \omega . \quad (99)$$

The combination $\square G$ is equivalent to the projector

$$\square G = \mathbb{1} - \sum_n \frac{\gamma_n}{(\int * \gamma_n \wedge \gamma_n)} \int * \gamma_n \wedge . \quad (100)$$

Here the γ_n form a basis of harmonic p -forms. For scalars on a compact manifold there is only one harmonic form, the constant, and for 0-forms the above combination reduces to

$$\square G(x, x') = \delta^D(x - x') - \frac{1}{\text{Vol}} . \quad (101)$$

⁴Recall that we use the convention that the Dirac-delta function in curved space obeys $\int dx \sqrt{g(x)} f(x) \delta(x - x_i) = f(x_i)$.

For one dimension this equals the r.h.s. of (97)

Particularly in the case of tori, one occasionally imposes the boundary conditions signifying the compactness of the space by introducing image charges on the covering space. The torus is considered as the quotient \mathbb{R}^d/Λ_d of \mathbb{R}^d by the lattice Λ_d . The Green's function on \mathbb{R}^d/Λ_d is then a sum of regular \mathbb{R}^d Green's functions obeying

$$\square G_{\mathbb{R}^d}(x, x') = \delta_{Dirac}^D(x - x') \quad (102)$$

such that $G_{\mathbb{R}^d/\Lambda_d}$ and its derivative is periodic, i.e.

$$G_{\mathbb{R}^d/\Lambda_d}(x, x') = \sum_{\vec{n}} G_{\mathbb{R}^d}(x, x' + \vec{n} \cdot \vec{e}) . \quad (103)$$

Here \vec{e} are the lattice vectors generating Λ_d .

The Laplacian acting on $G_{\mathbb{R}^d/\Lambda_d}(x, x')$ now gives

$$\square G_{\mathbb{R}^d/\Lambda_d}(x, x') = \sum_{\vec{n}} \delta_{Dirac}^D(x - x' + \vec{n} \cdot \vec{e}) . \quad (104)$$

The only relevant part of the r.h.s, however, is the term with $n = 0$ as both x, x' belong to a single fundamental region $x^i \in [0, e^i)$. The delta functions for other values of \vec{n} never contribute.

If one chooses to solve for the warp factor using such a Green's function, one would *ignore* the contribution from any constant terms on the r.h.s. This is evident from the preceding discussion. However, a consistency condition is required, namely, the total integral on the r.h.s. of the warp factor equation of motion must vanish. This just says that the total flux on the compact space is zero or that the anomaly cancels.

B. Heterotic-M theory on CY_3

The effective five-dimensional theory by compactifying Heterotic-M theory on a Calabi-Yau 3-fold was derived in [6]. Here, we will follow their discussion and notation. We will consider the simplest case, in which there are no 5-branes, and the low energy degrees of freedom include only gravity and the hypermultiplet containing the breathing mode V of the Calabi-Yau (the “universal” solution in [6]). With the standard embedding of the spin connection in one of the E_8 's, the effective action becomes [6]

$$\begin{aligned} 2\kappa_5^2 S_5 = & - \int_{M_5} \sqrt{-g} \left[R + \frac{1}{2} V^{-2} g^{55} V'^2 + \frac{1}{3} V^{-2} \alpha^2 \right] \\ & + 2\sqrt{2} \int_{M_4^{(1)}} \sqrt{-g} V^{-1} \alpha - 2\sqrt{2} \int_{M_4^{(2)}} \sqrt{-g} V^{-1} \alpha , \end{aligned} \quad (105)$$

where κ_5 is the five-dimensional gravitational coupling. The constant α is given by

$$\alpha = -\frac{1}{8\sqrt{2}\pi v} \left(\frac{\kappa}{4\pi} \right)^{2/3} \int_{CY_3} \omega \wedge \text{tr} R \wedge R , \quad v = \int_{CY_3} \sqrt{g_{CY_3}} , \quad (106)$$

where ω is the Kahler form on the CY_3 . The metric is warped to

$$ds^2 = a^2(y)\eta_{\mu\nu}dx^\mu dx^\nu + b^2(y)dy^2 . \quad (107)$$

The solution to the field equations is:

$$\begin{aligned} a &= a_0 H^{1/2} , \\ b &= b_0 H^2 , \quad H = \frac{\sqrt{2}}{3}\alpha|y| + c_0 \equiv c|y| + c_0 , \\ V &= b_0 H^3 , \end{aligned} \quad (108)$$

where a_0 , b_0 and c_0 are integration constants.

The Einstein equation linearized in the fluctuations give:

$$-\frac{1}{2}h''_{\mu\nu} - \frac{1}{2}\left(4\frac{a'}{a} - \frac{b'}{b}\right)h'_{\mu\nu} - \left(3cb_0H - 3\frac{c}{H}\right)\left(\delta(y) - \delta(y - \pi\rho)\right)h_{\mu\nu} = \frac{1}{2}\left(\frac{b_0}{a_0}\right)^2 H^3 h_{\mu\nu,\lambda\lambda} . \quad (109)$$

Let us first consider the equation in the bulk. It is easy to see that the above equation can be written as the covariant wave equation (39). As in Section 4, we expand $h_{\mu\nu}(x, y) = \hat{h}_{\mu\nu}(x)\psi(y)$ with $\square_x \hat{h}_{\mu\nu}(x) = m^2 \hat{h}_{\mu\nu}(x)$. The properly normalized wavefunction is

$$\Psi(y) = [e^{-2A(y)}\sqrt{g(y)}]^{1/2}\psi(y) = ab^{1/2}\psi(y) . \quad (110)$$

For massless graviton,

$$\Psi(y) \sim ab^{1/2} \sim H^{3/2} . \quad (111)$$

Therefore, the wavefunction $\Psi(y)$ vanishes at the singularity where $H(y)$ is zero.

The compactification scale of the M theory direction is usually taken to be slightly larger than that of the Calabi-Yau (from gauge and gravitational unification [2]). Therefore, there is a regime in which the theory is five-dimensional, and the wavefunction of some of the low-lying massive KK gravitons can still be described by the above wave equation

$$-\frac{1}{2}\psi''(y) - \left(3cb_0H - 3\frac{c}{H}\right)\left(\delta(y) - \delta(y - \pi\rho)\right)\psi(y) = \frac{1}{2}m^2\left(\frac{b_0}{a_0}\right)^2 H^3 \psi(y) . \quad (112)$$

Let us rewrite the above equation such that m^2 becomes the eigenvalue of a Schrodinger equation. Define a new variable $u = (2/5) c^{-1} (b_0/a_0) H^{5/2}$ and hence $du = (b_0/a_0)H^{3/2}dy$. In terms of this variable, the linearized equation becomes

$$-\frac{1}{2}\ddot{\psi} - \frac{3c}{4}\left(\frac{a_0}{b_0}\right)H^{-5/2}\dot{\psi} - 3a_0c\left(\frac{1}{H^{1/2}} - \frac{1}{b_0H^{5/2}}\right)\left(\delta(u - u_1) - \delta(u - u_2)\right)\psi = \frac{1}{2}m^2\psi , \quad (113)$$

where the dot denotes the derivative with respect to u . The locations u_1 and u_2 are defined by $u = u_i$ when $y = 0$ and $\pi\rho$ respectively.

Finally, to eliminate the first derivative term in the above equation. Define $\psi = H^s \hat{\psi}$. It is easy to see that first order derivatives of $\hat{\psi}$ do not appear if $s = -3/4$. The function $\hat{\psi}$ satisfies the following equation

$$-\frac{1}{2}\frac{d^2}{du^2}\hat{\psi}(u) + \mathcal{V}(u)\hat{\psi}(u) = \frac{1}{2}m^2\hat{\psi}(u) , \quad (114)$$

where

$$\begin{aligned}\mathcal{V}(u) &= -\frac{21}{32} \left(\frac{a_0}{b_0}\right)^2 \frac{c^2}{H^5} - \left(\frac{a_0}{b_0}\right) \frac{c}{H^{5/2}} \left(3b_0 H^2 - \frac{15}{4}\right) \left(\delta(u - u_1) - \delta(u - u_2)\right) \\ &= -\frac{21}{200u^2} - \frac{2}{5u} \cdot \left[3b_0 \left(\frac{5c a_0}{2 b_0} u\right)^{4/5} - \frac{15}{4}\right] \left(\delta(u - u_1) - \delta(u - u_2)\right) .\end{aligned}$$

The potential is not of the “volcano” type [8]. In particular, we have seen that gravity is not localized.

We have seen that the wavefunction of the massless graviton can be deduced without having to solve (114) since it is clear from (41) that $\psi = \text{constant}$ is a solution. The massive modes can be found by solving (41), or equivalently (114). Let us focus on the latter, as it allows us to compare our solution with that of [8]. Let $\hat{\psi} = u^{1/2} f$, the Schrodinger equation can be written as

$$v^2 \frac{\partial^2 f}{\partial v^2} + v \frac{\partial f}{\partial v} + (v^2 - \frac{1}{25}) f(v) = 0 , \quad (115)$$

where $v = m(u + u_0)$. The solution is simply

$$f = k_1 J_{1/5}(m(u + u_0)) + k_2 J_{-1/5}(m(u + u_0)) , \quad (116)$$

where k_1, k_2 are constants which can be determined by the matching conditions at the delta function sources. The boundary conditions at the two delta sources also imply that the masses m of the KK modes are quantized in units of $1/\rho$.

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